Finite Element Time Domain Method using Laguerre Polynomials

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ABSTRACT — **In this work, we present a numerical method to obtain an unconditionally stable solution for the finite element method in time domain (FETD) for two-dimensional TE_z case. Our method does not utilize the customary marching-on in time solution method often used to solve a hyperbolic partial differential equation. Instead we solve the time domain wave equation by expressing the transient behaviors in terms of weighted Laguerre polynomials. By using these causal orthonormal basis functions for the temporal variation, the time derivatives can be handled analytically. To verify our method, we apply it to two-dimensional parallel plate waveguide and compare the result to that of the conventional FETD using the Newmark-Beta method.**

I. INTRODUCTION

In recent times, the finite element method in time domain (FETD) has been introduced to analyze transient electromagnetic problems [1]-[3]. By introducing triangular or tetrahedral elements in two or three-dimensional problem, it is easy to apply the FETD method to highly complex shaped models. And by using the Newmark-Beta method [4], one can obtain an unconditionally stable FETD formulation. By introducing the Newmark-Beta method, although one can eliminate the limitation of time step, the larger value of the time step causes larger numerical error.

In this paper, we propose a new unconditionally stable solution procedure for the FETD method for the two-dimensional TE_z case using weighted Laguerre polynomials as temporal basis and testing functions. By introducing the temporal testing procedure, instead of the marching-on in time technique, we introduce the marching-on in order of the temporal functions. Therefore, we can obtain the unknown coefficients for the basis functions from the 0^{th} order to the N_L^{th} order by solving recursively the proposed new FETD. And also, the proposed method produces a same banded sparse system matrix as the conventional FETD method, which is independent of the order of testing functions. So, we need to assemble this sparse system matrix only once as like the conventional FETD method with the Newmark-Beta method.

The paper is organized as follows. Section 2 presents the conventional FETD formulation and the proposed FETD formulation using weighted Laguerre polynomials. The numerical results are presented in Section 3. Some conclusions are given in Section 4.

II. FINITE ELEMENT IN TIME DOMAIN

A. FETD with TE_z case

For a simple and lossless media, the time domain vector wave equation is given by

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}}{\partial t} \qquad (1)$$

where c is the velocity of the light in media and μ is the permeability. Using a vector testing function $\mathbf{W}_i$, we have the weak formulation of (1) as follows

$$\int_\Omega \left((\nabla \times \mathbf{W}_i)\cdot(\nabla \times \mathbf{E}) + \mathbf{W}_i \cdot \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \right) d\Omega + \int_\Gamma \mathbf{W}_i \cdot (\hat{n} \times \nabla \times \mathbf{E}) d\Gamma = -\int_\Omega \mu \mathbf{W}_i \cdot \frac{\partial \mathbf{J}}{\partial t} d\Omega \qquad (2)$$

where Ω is bounded computational domain by boundary gamma and Γ denotes the absorbing boundary. In this paper, all of domain is divided into triangular elements. Using the Galerkin's method, the electric field variable throughout omega is expanded in terms of N vector basis functions which are same as the above vector testing function as follows

$$\mathbf{E}(\mathbf{r},t) = \sum_{j=1}^{N} E_j(t)\mathbf{W}_j(\mathbf{r}) \qquad (3)$$

where $\mathbf{W}_j$ is the linear edge-based vector basis function [5] and E_j is its corresponding temporal coefficient. To truncate the computational domain Ω numerically, we should apply an absorbing boundary condition on Γ. In this paper, we use the 1st order ABC given by

$$\hat{n}\times\nabla\times\mathbf{E}+\frac{1}{c}\hat{n}\times\hat{n}\times\frac{\partial\mathbf{E}}{\partial t}=0 \tag{4}$$

where $\hat{n}$ denotes the outward unit vector normal to the absorbing boundary Γ. Using some vector identities and inserting (4) into (2), we have the matrix equation of (2)

$$[M]\{\ddot{E}\}+[B]\{\dot{E}\}+[K]\{E\}=\{Q\} \tag{5}$$

where upper dot denotes the derivative with respect to time. And the element matrices are given by

$$M_{ij}^{e}=\frac{1}{c^2}\int_{\Omega^e}\mathbf{W}_i\cdot\mathbf{W}_j d\Omega^e \tag{6a}$$

$$B_{ii}^{e}=\int_{\Gamma^e}(\hat{n}\times\mathbf{W}_i)\cdot(\hat{n}\times\mathbf{W}_i)d\Gamma^e \tag{6b}$$

$$K_{ij}^{e}=\int_{\Omega^e}(\nabla\times\mathbf{W}_i)\cdot(\nabla\times\mathbf{W}_j)d\Omega^e \tag{6c}$$

$$Q_i^e=-\mu\int_{\Omega^e}\dot{\mathbf{J}}\cdot\mathbf{W}_i d\Omega^e \tag{6d}$$

Based on the Newmark-Beta method, (5) is approximated as

$$\begin{aligned}&\left[\frac{M}{\Delta t^2}\right]\left\{E^{n+1}-2E^n+E^{n-1}\right\}+\left[\frac{B}{2\Delta t}\right]\left\{E^{n+1}-E^{n-1}\right\}+\\&[K]\left\{\beta E^{n+1}+(1-2\beta)E^n+\beta E^{n-1}\right\}\\&=\left\{\beta Q^{n+1}+(1-2\beta)Q^n+\beta Q^{n-1}\right\}\end{aligned} \tag{7}$$

where E^n is the discrete-time representation of E. In this paper, we use β=1/4. By solving (7) in temporal sequence, one can obtain the unconditionally stable solution. That is, by introducing the Newmark-Beta method, we can eliminate limitation of time step Δt in (7). However, the larger value of the time step results in larger numerical error. In the following part, we propose a new FETD algorithm, which users weighted Laguerre polynomials as temporal basis functions.

B. FETD with Weighted Lagerrre Polynomials

Consider the set of polynomials defined by

$$L_p(t)=\frac{e^t}{p!}\frac{d^p}{dt^p}\left(t^p e^{-t}\right),\ \text{for } p\geq 0,\ t\geq 0 \tag{8}$$

These are Laguerre polynomials of order p that are causal, which means that they exist for $t\geq 0$. These polynomials satisfy the following recursive relationship

$$L_0(t)=1,\ L_1(t)=1-t$$

$$pL_p(t)=(2p-1-t)L_{p-1}(t)-(p-1)L_{p-2}(t) \tag{9}$$

where $p\geq 2,\ t\geq 0$. The Laguerre polynomials are orthogonal with respect to the weighting function e^{-t}, given by

$$\int_0^\infty e^{-t}\cdot L_p(t)L_q(t)dt=\delta_{pq} \tag{10}$$

where δ_{pq} is the Kronecker delta for $p=q$ and zero otherwise. Therefore, an orthonormal set of basis functions $\{\varphi_0,\varphi_1,\varphi_2,\ldots\}$ can derived from (10) through the representation

$$\varphi_p(t,s)=e^{-s\cdot t/2}L_p(s\cdot t) \tag{11}$$

where $s>0$ is a time scale factor. Note that these functions are absolutely convergent to zero as $t\to\infty$. Hence arbitrary functions spanned by these basis functions are also absolutely convergent to zero as $t\to\infty$. These basis functions are also orthogonal with respect to the scaled time variable $\bar{t}$ as

$$\langle\varphi_p,\varphi_q\rangle\equiv\int_0^\infty\varphi_p(\bar{t})\cdot\varphi_q(\bar{t})d\bar{t}=\delta_{pq} \tag{12}$$

where $\bar{t}=s\cdot t$ is the scaled time. Since the real time scale is quite small, in order to use the above basis functions properly, one should transform the real time scale using an appropriate scale factor. These orthogonal functions can approximate causal electromagnetic responses quite well. By controlling the time scale factor s, the support provided by the expansion can be increased or decreased. Basis functions of order 0 to 4 are plotted in Fig. 1. As can be seen, the functions given by (11) are causal and convergent as $t\to\infty$.

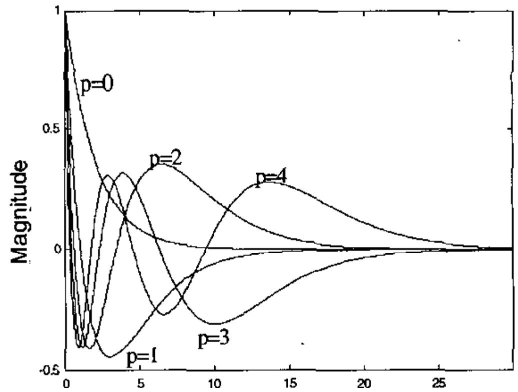


Fig.1: Weighted Laguerre polynomials with different orders

Using these temporal basis functions, the temporal coefficients in (3) can be expanded as

$$E_j(t)=\sum_{p=0}^{\infty}E_{j,p}\varphi_p(\bar{t}) \tag{13}$$

In [6], we can show that the 1st and 2nd derivative of $E_j(t)$ with respect to time t are

$$\dot{E}_j(t) = s\sum_{p=0}^{\infty}\left(0.5E_{j,p} + \sum_{k=0}^{p-1}E_{j,k}\right)\varphi_p(\bar{t}) \quad (14)$$

$$\ddot{E}_j(t) = s^2\sum_{p=0}^{\infty}\left(0.25E_{j,p} + \sum_{k=0}^{p-1}(p-k)E_{j,k}\right)\varphi_p(\bar{t}) \quad (15)$$

Inserting (13), (14), and (15) into (5), then we have

$$s^2[M]\left\{\sum_{p=0}^{\infty}\left(0.25E_{j,p} + \sum_{k=0}^{p-1}(p-k)E_{j,k}\right)\varphi_p(\bar{t})\right\} + s[B]\left\{\sum_{p=0}^{\infty}\left(0.5E_{j,p} + \sum_{k=0}^{p-1}E_{j,k}\right)\varphi_p(\bar{t})\right\} + [K]\left\{\sum_{p=0}^{\infty}E_{j,p}\varphi_p(\bar{t})\right\} = \{Q(t)\} \quad (16)$$

Multiplying both sides of (16) by $\varphi_q(\bar{t})$ and integrating over $\bar{t} = [0,\infty)$, and using the orthogonality relation (12), then we get

$$([K] + 0.5s[B] + 0.25s^2[M])\{E_q\} = -s[B]\left\{\sum_{k=0}^{q-1}E_k\right\} - s^2[M]\left\{\sum_{k=0}^{q-1}(q-k)E_k\right\} + \{Q_q\} \quad (17)$$

where

$$Q_q = \int_0^{\infty} Q(t)\varphi_q(\bar{t})d\bar{t} \quad (18)$$

In our method, the time step is used only to calculate the Laguerre coefficients due to the excitation in (18) at the beginning of the computation. Therefore, one can choose small Δt to evaluate (18) accurately, which does not influence the computing time.

C. Choice of The Number of Basis Functions

It is assumed that the signal that we are interested in characterizing is practically bandlimited up to a frequency B. In addition, we are also interested in generating the same signal in the time domain upto the time duration T_f. Then, we represent the real time signal $P(t)$ by a Fourier series,

$$P(t) = \sum_k C_k e^{jk\omega_0 t} \quad (19)$$

where $\omega_0 = 2\pi/T_f$. Since $P(t)$ is real, $C_k^* = C_{-k}$ where * means conjugate transpose. If $P(t)$ is bandlimited to B Hertz, then the value of u can be fixed by

$$-B \le \frac{k}{T_f} \le B \quad (20)$$

Therefore we have

$$P(t) = \sum_{k=-BT_f}^{BT_f} C_k e^{jk\omega_0 t} \quad (21)$$

In (21), there are $2BT_f+1$ terms in the expansion of $P(t)$. Hence, the minimum number of temporal basis functions is

$$N_L = 2BT_f + 1 \quad (22)$$

In order to obtain an accurate solution, therefore, one should solve (17) recursively at least N_L times. Therefore, if we wan to observe the transient response at a spatial location due to an incident field of bandwidth $2B$, then we need at least $2BT_f + 1$ terms of the Laguerre series to completely characterize that temporal waveform of duration T_f and bandwidth $2B$ irrespective of its shape.

III. NUMERICAL EXAMPLE

In this section, two-dimensional parallel plate waveguide for the TE_z case is tested to validate our method as shown in Fig. 2. In this paper, we use the following sinusoidally modulated Gaussian pulse as an input electric current profile.

$$J_y(t) = \exp\left(-\left(\frac{t-T_c}{T_d}\right)^2\right)\sin(2\pi f_c(t-T_c)) \quad (23)$$

where

$$T_d = \frac{1}{2f_c},\ T_c = 3T_d \quad (24)$$

In this paper, we use $f_c = 1$ GHz. And we choose $T_f = 10$ ns and $B = 5$ GHz. Inserting T_f and B into (22), we can evaluate the number of the weighted Laguerre polynomial functions, and we choose $N_L = 150$, which is sufficient to approximate the various responses. And the time scale factor is $s = 6.07\times10^{10}$.

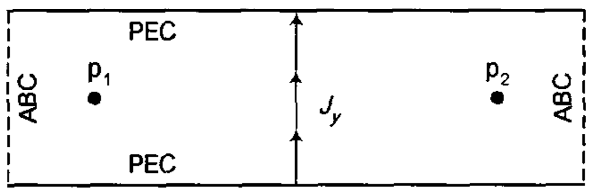
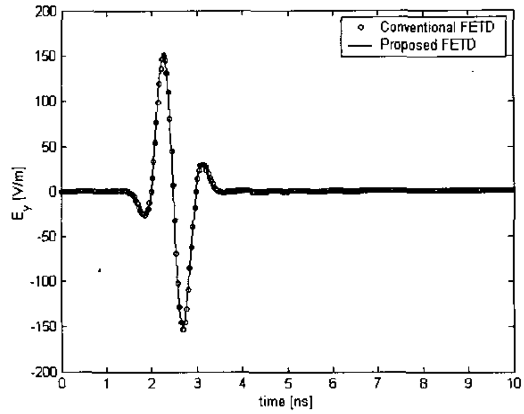


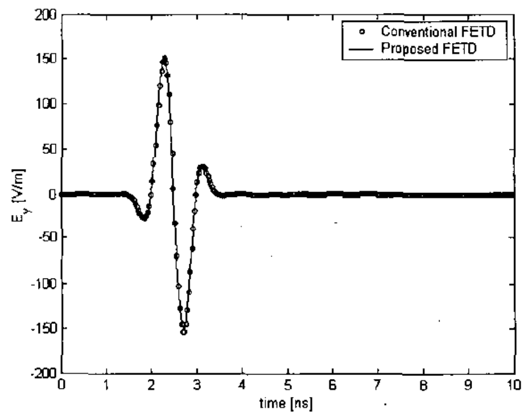
Fig. 2: A two-dimensional parallel plate waveguide model

The computational domain was divided into 2000 triangular elements and 3110 edges. We put two measurement points, p_1 and p_2, which are the same distant from the source. To truncate the computational domain, we used the 1st order absorbing boundary condition at the x-directional terminals of the domain. Fig.

3 shows the *y*-components of the electric fields at the measurement points. The agreement between the conventional FETD method and the proposed method is very good. In the conventional FETD method, there are 300 times steps.



(a) at p_1



(b) at p_2

Fig. 3: Transient electric fields of the y-component

IV. CONCLUSIONS

A new unconditionally stable solution for the time domain finite element method has been proposed for the two-dimensional TE_z case. We utilize a marching-on in order method to solve the proposed FETD method with weighted Laguerre polynomials. Using the temporal basis functions, the temporal derivatives can be handled analytically. Also, transient fields obtained by the proposed method are unconditionally stable regardless of the time step size. Moreover, the agreement between the results obtained using the proposed method and the conventional method is very good.

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